Problem 1 *Hyperbolic functions* (4 points)
The hyperbolic functions are defined by $\cosh(x) := \frac{1}{2}(e^x + e^{-x})$, $\sinh(x) := \frac{1}{2}(e^x - e^{-x})$ and $\tanh(x) := \frac{\sinh(x)}{\cosh(x)}$ for all $x \in \mathbb{R}$.

a) Prove that

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} \quad \text{and} \quad \sinh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$$

hold for every $x \in \mathbb{R}$, and determine the radius of convergence of these series.

b) Show that

$$\cosh^2(x) - \sinh^2(x) = 1$$

holds for all $x \in \mathbb{R}$.

c) Show the addition theorems

$$\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)\,,$$
$$\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)\,,$$

for all $x, y \in \mathbb{R}$.

d) By using (c), show that

$$\sinh(2x) = 2\sinh(x)\cosh(x) = \frac{2\tanh(x)}{1 - \tanh^2(x)}$$

holds for all $x \in \mathbb{R}$.

Solutions

(a)

$$e^x = \sum_{j=0}^{\infty} \frac{1}{j!} x^j = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$$

$$e^{-x} = \sum_{j=0}^{\infty} \frac{1}{j!} (-x)^j = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k} - \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$$

$$\Rightarrow \cosh(x) = \frac{1}{2}(e^x + e^{-x}) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$$

$$\sinh(x) = \frac{1}{2}(e^x - e^{-x}) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} x^{2k+1}$$

(b) $$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = \underline{1}$$

(c) $$\sinh(x)\cosh(y) + \cosh(x)\sinh(y) = \frac{1}{4}\left(e^{x} - e^{-x}\right)\left(e^{y} + e^{-y}\right) + \frac{1}{4}\left(e^{x} + e^{-x}\right)\left(e^{y} - e^{-y}\right)$$

$$= \frac{1}{4}\left(e^{x+y} - e^{y-x} + e^{x-y} - e^{-x-y} + e^{x+y} + e^{y-x} - e^{x-y} - e^{-x-y}\right)$$

$$= \frac{1}{2}\left(e^{x+y} - e^{-(x+y)}\right) = \sinh(x+y)$$

$$\cosh(x)\cosh(y) + \sinh(x)\sinh(y) = \frac{1}{4}\left(e^{x} + e^{-x}\right)\left(e^{y} + e^{-y}\right) + \frac{1}{4}\left(e^{x} - e^{-x}\right)\left(e^{y} - e^{-y}\right)$$

$$= \frac{1}{4}\left(e^{x+y} + e^{y-x} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{y-x} - e^{x-y} + e^{-x-y}\right)$$

$$= \frac{1}{2}\left(e^{x+y} + e^{-(x+y)}\right) = \cosh(x+y)$$

(d) $$\sinh(2x) = \sinh(x+x) \overset{(c)}{=} 2\sinh(x)\cosh(x) \quad \checkmark$$

$$\frac{2\tanh(x)}{1 - \tanh^2(x)} = \frac{2\,\frac{\sinh(x)}{\cosh(x)}}{1 - \frac{\sinh^2(x)}{\cosh^2(x)}} = \frac{2\sinh(x)\cosh(x)}{\underbrace{\cosh^2(x) - \sinh^2(x)}_{=1}}$$

$$= 2\sinh(x)\cosh(x) \quad \checkmark$$