Problem 3 *Radius of convergence of power series* (4 points)
Determine the radius of convergence of the following power series:

a) $\sum_{k=1}^{\infty} 3^{-k} x^k$

b) $\sum_{k=1}^{\infty} k!(x-3)^k$

c) $\sum_{k=1}^{\infty} \binom{2k}{k} (x+5)^k$

d) $\sum_{k=1}^{\infty} (-1)^n \cdot \frac{4^k + e^k}{k^3} (x-11)^k$

e) $\sum_{k=1}^{\infty} k^3 3^{-k!} (x-3)^{k!}$

f) $\sum_{k=1}^{\infty} 3^{-k} x^{3k}$

g) $\sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^{k^2} x^k$

h) $\sum_{k=1}^{\infty} k^{\sqrt{k}} x^k$

Solutions

(a) $R^{-1} = \limsup_{k\to\infty} \sqrt[k]{|a_k|} = \frac{1}{3} \Rightarrow R = 3$

(b) $\left|\frac{a_k}{a_{k+1}}\right| = \frac{k!}{(k+1)k!} = \frac{1}{k+1} \xrightarrow{k\to\infty} 0 \Rightarrow R = 0$

(c) $a_k = \frac{(2k)!}{k!\,k!}$, $a_{k+1} = \frac{(2(k+1))!}{(k+1)!\,(k+1)!} = \frac{(2k+2)(2k+1)\cdot(2k)!}{(k+1)\cdot k!\cdot(k+1)\cdot k!}$

$$\left|\frac{a_k}{a_{k+1}}\right| = \frac{(k+1)\cdot(k+1)}{(2k+2)(2k+1)} = \frac{k^2+2k+1}{4k^2+6k+2} = \frac{1+\frac{2}{k}+\frac{1}{k^2}}{4+\frac{6}{k}+\frac{2}{k^2}}$$

$$\xrightarrow{k\to\infty} \frac{1}{4} \Rightarrow R = \frac{1}{4}$$

(d) $|a_k|^{\frac{1}{k}} = \left(\frac{4^k + e^k}{k^3}\right)^{\frac{1}{k}} = \underbrace{\frac{1}{\sqrt[k]{k^3}}}_{\text{Convergent to } 1} (4^k + e^k)^{\frac{1}{k}}$, $\left(\begin{array}{c}\text{Using:} \\ 1 \le e \le 4\end{array}\right)$

We have $(4^k)^{\frac{1}{k}} \le (4^k + e^k)^{\frac{1}{k}} \le (4^k + 4^k)^{\frac{1}{k}}$

By sandwich theorem we get:

$$\lim_{k \to \infty} |a_k|^{\frac{1}{k}} = 4 \quad \Rightarrow \quad R = \frac{1}{4}$$

(e) Here the power series is:

$$\sum_{k=1}^{\infty} k^3\, 3^{-k!} (x-3)^{k!} = \sum_{n=1}^{\infty} a_n (x-3)^n$$

with $a_n = \begin{cases} k^3 \cdot 3^{-n} & \text{if there is a } k \in \mathbb{N} \text{ with } n = k! \\ 0 & \text{else} \end{cases}$

$$|a_n|^{\frac{1}{n}} = \underbrace{(k^3)^{\frac{1}{n}}}_{\text{between } (1)^{\frac{1}{n}} \text{ and } (n)^{\frac{1}{n}}} \cdot 3^{-1} \xrightarrow{n \to \infty} \frac{1}{3} \quad \Rightarrow \quad R = 3$$

(f) $\sum_{k=1}^{\infty} 3^{-k} x^{3k} = \sum_{n=1}^{\infty} a_n x^n$ with $a_n = \begin{cases} 3^{-\frac{n}{3}} & , n = 3k \text{ for a } k \\ 0 & , \text{else} \end{cases}$

$$\limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = 3^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{3}} \quad \Rightarrow \quad R = \sqrt[3]{3}$$

(g) $|a_k|^{\frac{1}{k}} = \left(1 + \frac{1}{k}\right)^k \xrightarrow{k \to \infty} e \quad \Rightarrow \quad R = \frac{1}{e}$

(h) $|a_k|^{\frac{1}{k}} = k^{\frac{1}{\sqrt{k}}} =: b_k$, where $(b_k)_{k\in\mathbb{N}}$ is mon. decreasing for $k > e^2$,

and bonded from below. This means that it is convergent.

The subsequence $(b_{n^2})_{n\in\mathbb{N}}$ has the same limit:

$$(n^2)^{\frac{1}{n}} = \left(\sqrt[n]{n}\right)^2 \xrightarrow[\text{Sheet 2}]{n\to\infty} 1^2 \quad \Rightarrow \quad \underline{R = 1}$$

Remark: It is good enough, when they have seen that the subsequence is convergent. They do not have to show that $(b_k)_{k\in\mathbb{N}}$ is eventually decreasing. I will show the solution now in full detail:

First note that for $x > 0$:

(cf. lecture chapter 5)

$$\frac{e^{\frac{x}{2}}}{x} \geq \frac{1}{x}\left(1 + \frac{x}{2} + \frac{x^2}{4}\right) = \frac{1}{x} + \frac{1}{2} + \frac{x}{4} \xrightarrow{x\to\infty} \infty .$$

For each positive sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \to \infty$, we have:

$$x_k \cdot e^{-\frac{x_k}{2}} \xrightarrow{k\to\infty} 0. \qquad (*)$$

Back to the sequence $(b_k)_{k\in\mathbb{N}}$:

$$b_k = k^{\frac{1}{\sqrt{k}}} = e^{\log\left(k^{\frac{1}{\sqrt{k}}}\right)} = e^{\overbrace{\frac{1}{\sqrt{k}}\log(k)}^{=:c_k}}$$

Since exp is a mon. increasing function, we look just at the sequence $(c_k)_{k\in\mathbb{N}}$:

possible by chapter 5

$$c_k = \frac{\log(k)}{\sqrt{k}} = \frac{\log(e^{x_k})}{\sqrt{e^{x_k}}} \quad \text{with} \quad e^{x_k} = k$$

$$= x_k \cdot e^{-\frac{x_k}{2}} \xrightarrow{k\to\infty} 0 \quad \text{by} \quad (*) .$$

By continuity of exp, we get:

$$\lim_{k\to\infty} b_k = e^0 = 1. \quad \Rightarrow \quad \underline{R = 1}$$