Problem 1 *Derivatives of elementary functions* (4 points)

a) Use the power series for exp, sin, cos, sinh and cosh to calculate its derivatives.

b) Use the differentiation of inverse functions to calculate the derivatives of arcsin, arccos, arctan, arsinh and arcosh.

Solutions

(a) Use $\frac{d}{dx} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=1}^{\infty} a_k \cdot k \cdot x^{k-1}$

We get $\exp' = \exp$, $\sin' = \cos$, $\cos' = -\sin$, $\sinh' = \cosh$, $\cosh' = \sinh$.

(b) Use $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$ for y in domain of definition of f^{-1}.

For $f(x) = \sin(x)$: $f'(x) = \cos(x)$, $f^{-1}(y) = \arcsin(y)$

$$f'(f^{-1}(y)) = \cos(\arcsin(y))$$

using $\cos^2 + \sin^2 = 1$ and (*)

$$= \sqrt{1 - \sin^2(\arcsin(y))}$$

$$= \sqrt{1 - y^2}$$

$$\Rightarrow \arcsin'(y) = \frac{1}{\sqrt{1-y^2}}$$

Remark: $\arcsin(y) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \cos(\arcsin(y)) \in [0,1]$ (*)
(non-negative)

For $f(x) = \cos(x)$: $\quad f'(x) = -\sin(x)$, $\quad f^{-1}(y) = \arccos(y)$

$$f'(f^{-1}(y)) = -\sin(\arccos(y))$$

$$= -\sqrt{1-\cos^2(\arccos(y))}$$

using $\cos^2+\sin^2=1$ and (**)

$$= -\sqrt{1-y^2}$$

$$\Rightarrow \arccos'(y) = -\frac{1}{\sqrt{1-y^2}}$$

Now we have $\arccos(y) \in [0,\pi] \Rightarrow \sin(\arccos(y)) \in [0,1]$ (**)

Analogously, we calculate the derivatives for arsinh and arcosh, while using $\cosh^2 - \sinh^2 = 1$.

We get $\quad \operatorname{arcosh}'(y) = \dfrac{1}{\sqrt{y^2-1}}$, $\quad \operatorname{arsinh}'(y) = \dfrac{1}{\sqrt{y^2+1}}$

For arctan, one has to calculate the derivative of tan, first.

$$\tan'(x) = \frac{\cos\cdot\cos - \sin\cdot(-\sin)}{\cos^2} = 1 + \frac{\sin^2(x)}{\cos^2(x)} = 1+\tan^2(x).$$

For $f(x) = \tan(x)$: $\quad f'(x) = 1+\tan^2(x)$, $\quad f^{-1}(y) = \arctan(y)$

$$f'(f^{-1}(y)) = 1 + \tan^2(\arctan(y))$$

$$= 1 + y^2$$

$$\Rightarrow \arctan'(y) = \frac{1}{1+y^2}$$