Problem 3 *Continuity, differentiability and continuous differentiability* (4 points)
For $k = 1, 2, 3$ define

$$f_k \colon \mathbb{R} \to \mathbb{R},\ f_k(x) := \begin{cases} x^k \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show the following:

a) f_1, f_2 and f_3 are continuous functions.

b) f_1 is differentiable on $\mathbb{R} \setminus \{0\}$ but not in 0.

c) f_2 is differentiable on $\mathbb{R}$ but its derivative f' is not continuous in 0.

d) f_3 is differentiable on $\mathbb{R}$ and its derivative f' is continuous function on $\mathbb{R}$.

Hint: For the special case $x = 0$, one has to make use of the definitions of differentiability and continuity.

Solutions

(a) f_k is continuous for all $x \neq 0$ since it is a product of polynomial and sin-function (both continuous!).

For $x_0 = 0$, we see for $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}\setminus\{0\}$ with $x_n \to 0$.

$$|f_k(x)| = |x_n^k| \cdot \underbrace{\left|\sin\left(\frac{1}{x_n}\right)\right|}_{\leq 1} \leq |x_n|^k \xrightarrow{n\to\infty} 0.$$

(b) f_1 is differentiable for all $x \neq 0$ since it is a product and composition of differentiable functions (product + chain rule).

For $x_0 = 0$:
$$\frac{f(h) - f(0)}{h} = \sin\left(\frac{1}{h}\right) \quad \text{for all } h \neq 0.$$

The limit $h \to 0$ of $\sin\left(\frac{1}{h}\right)$ doesn't exist. $\Rightarrow$ not differentiable.

(c) f_2 is differentiable for all $x \neq 0$ since it is a product and composition of differentiable functions (roduct + chain rule).

For $x_0 = 0$:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) \overset{\text{as in (a)}}{=} 0$$

$\Rightarrow$ f_2 is also differentiable in $x_0 = 0$, with $f_2'(0) = 0$.

In summary, we get:

$$f_2'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Here, $\lim_{x \to 0} f_2'(x)$ does not exist $\Rightarrow$ not continuous in $x_0 = 0$.

(d) f_3 is differentiable for all $x \neq 0$ since it is a product and composition of differentiable functions (roduct + chain rule).

For $x_0 = 0$:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h^2 \sin\left(\frac{1}{h}\right) \overset{\text{as in (a)}}{=} 0$$

$\Rightarrow$ f_3 is also differentiable in $x_0 = 0$, with $f_3'(0) = 0$.

In summary, we get:

$$f_3'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right) & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases}$$

Here, $\lim_{x \to 0} f_3'(x) = 0$ (reasoning like in (a)).

$\Rightarrow$ f_3' is continuous in $x_0 = 0$ and
f_3' is continuous for all $x_0 \neq 0$ since it is a combination of continuous functions.