Problem 1 *Calculate some limits* (4 points)
Determine the following limits:

a) $\lim_{x\to\infty} x^\alpha e^{-x}$, $\alpha > 0$

b) $\lim_{x\to\infty} \frac{\ln(x)}{x^\alpha}$, $\alpha > 0$

c) $\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x$ for $a \in \mathbb{R}$

d) $\lim_{x\to 0} \frac{e^x - \sin(x) - 1}{x^2}$

e) $\lim_{x\to\infty} \frac{\sinh(x)}{\cosh(x)}$

f) $\lim_{x\to 0} \frac{x-\sin(x)}{x^3}$

g) $\lim_{x\to 0} \frac{1-\cos(2x)}{1-\cos(x)}$

h) $\lim_{x\to 0} \frac{1-\frac{1}{3}x^2 - x\frac{\cos(x)}{\sin(x)}}{x^4}$

Solutions

Let $\alpha > 0$ be given. Choose $n \in \mathbb{N}$ such that $n-1 < \alpha \leq n$

Set $f(x) = x^\alpha$, $g(x) = e^x$,

$\Rightarrow f^{(k)}(x) = \alpha(\alpha-1)\cdot\ldots\cdot(\alpha-k+1)x^{\alpha-k}$, $g^{(k)}(x) = e^x$

Especially $\lim_{x\to\infty} f^{(n-1)}(x) = \lim_{x\to\infty} g^{(n-1)}(x) = \infty$ and

$$\lim_{x\to\infty} f^{(n)}(x) = \begin{cases} 0 & \alpha < n \\ \prod_{k=0}^{n-1}(\alpha-k) & \alpha = n \end{cases}, \quad \lim_{x\to\infty} g^{(n)}(x) = \infty$$

l'Hospital $\Rightarrow$

$$\lim_{x\to\infty} \frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} = \lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} = 0$$

$= \leftarrow$ apply now l'Hospital $(n-1)$ times more

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} x^\alpha e^{-x}$$

b) $\lim_{x\to\infty} \frac{\ln(x)}{x^\alpha} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0$

c) $\lim\limits_{x \to \infty} \ln\left(1 + \frac{a}{x}\right)^x = \lim\limits_{x \to \infty} x \cdot \ln\left(1 + \frac{a}{x}\right) = \lim\limits_{x \to \infty} \dfrac{\frac{1}{1+\frac{a}{x}} \cdot \left(-\frac{a}{x^2}\right)}{-\frac{1}{x^2}}$

$= a \Rightarrow \lim\limits_{x \to \infty} \left(1 + \frac{a}{x}\right)^x \underset{\substack{\uparrow \\ \text{continuity}}}{=} e^{\lim\limits_{x \to \infty} x \cdot \ln\left(1 + \frac{a}{x}\right)} = \underline{\underline{e^a}}$

d) $\lim\limits_{x \to 0} \dfrac{e^x - \sin(x) - 1}{x^2} = \lim\limits_{x \to 0} \dfrac{e^x - \cos(x)}{2x}$

$= \lim\limits_{x \to 0} \dfrac{e^x + \sin(x)}{2} = \dfrac{1}{2}$

Remark: One can also use power series but I wanted to use l'Hospital twice.

e) $\lim\limits_{x \to \infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \lim\limits_{x \to \infty} \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = 1$

Remark: This is a rather trivial example where l'Hospital will not help you.

f) $\lim\limits_{x \to 0} \dfrac{x - \sin(x)}{x^3} = \lim\limits_{x \to \infty} \dfrac{\frac{1}{6}x^3 - \sum\limits_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}}{x^3}$

$= \dfrac{1}{6} - \underbrace{\lim\limits_{x \to \infty} \sum\limits_{k=2}^{\infty} \dfrac{(-1)^k}{(2k+1)!} x^{2k-2}}_{=0} = \dfrac{1}{6}$

by making use of uniform convergence of the series around 0. I told them about that in the lecture. Nevertheless check if they really mention the argument.

g) $$\lim_{x \to 0} \frac{1-\cos(2x)}{1-\cos(x)} = \lim_{x \to 0} \frac{2\sin(2x)}{\sin(x)}$$

$$= 2 \lim_{x \to 0} \underbrace{\frac{\sin(2x)}{2x}}_{\to 1} \cdot \underbrace{\frac{x}{\sin(x)}}_{\to 1} \cdot 2 = \underline{\underline{4}}$$

Remark:
$$\lim_{x \to \infty} \frac{\sin(ax)}{x} = a$$
via l'Hospital.

h) $$\lim_{x \to 0} \frac{\sin x - \frac{1}{3}x^2 \sin x - x\cos x}{x^4 \sin x} =$$

$$\lim_{x \to 0} \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7) - \frac{1}{3}x^2\left(x - \frac{1}{6}x^3 + O(x^5)\right) - x\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6)\right)}{x^5 + O(x^7)}$$

$$= \lim_{x \to 0} \frac{\frac{1}{120}x^5 - \frac{1}{24}x^5 + \frac{1}{18}x^5}{x^5 + O(x^7)} = -\frac{1}{30} + \frac{1}{18} = \frac{10-6}{180} = \frac{4}{180} = \underline{\underline{\frac{1}{45}}}$$

Remark: Here the important idea is to shift $\sin(x)$ to the denominator so we can use power series.