Problem 2 *Higher derivatives and Leibniz formula* (4 points)
Let $n \in \mathbb{N}_0$, $I \subseteq \mathbb{R}$ be an open interval.

a) Assume $f, g \in C^n(I)$. Show that $fg \in C^n(I)$ and also

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)} .$$

Here, we denote $f^{(0)} := f$ and analogously for g and fg.

b) Calculate $h^{(2018)}$ for $h\colon \mathbb{R} \to \mathbb{R}$ with $h(x) := x^3 e^x$.

c) Assume that $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$. Show that

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{2h} .$$

Hint: Convince yourself that for each null sequence $(h_n)_{n\in\mathbb{N}}$, there are null sequences $(c_n)_{n\in\mathbb{N}}$ and $(d_n)_{n\in\mathbb{N}}$ such that

$$f(x_0 + h_n) = f(x_0) + h_n f'(x_0) + c_n h_n$$
$$f(x_0 - h_n) = f(x_0) - h_n f'(x_0) + d_n h_n$$

for all $n \in \mathbb{N}$ and use this fact.

Solutions

(a) Use induction and the product rule!

Induction step: $n \to n+1$

$$(fg)^{(n+1)} = \left((fg)^{(n)}\right)' \overset{\text{i.h.}}{=} \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\right)'$$

$$= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n-k+1)} g^{(k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k+1)}\right)$$

$$= \left(\sum_{k=0}^{n} \binom{n}{k} f^{(n+1-k)} g^{(k)} + \sum_{\ell=1}^{n+1} \binom{n}{\ell-1} f^{(n+1-\ell)} g^{(\ell)} \right.$$

$$= \binom{n}{0} f^{(n+1)} g^{(0)} + \sum_{k=1}^{n} \underbrace{\left(\binom{n}{k} + \binom{n}{k-1} \right)}_{\binom{n+1}{k}} f^{(n+1-\ell)} g^{(\ell)} + \binom{n}{n} f^{(0)} g^{(n+1)}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1-k)} g^{(\ell)} \qquad \square$$

(b)

$h(x) = f(x) \cdot g(x)$, $f(x) = e^x$, $g(x) = x^3$

$$h^{(2018)} = \sum_{k=0}^{2018} \binom{2018}{k} f^{(2018-k)} g^{(k)}$$

Since $g'(x) = 3x^2$, $g''(x) = 6x$, $g'''(x) = 6$, $g^{(4)}(x) = 0$,

we get:
$$h^{(2018)}(x) = e^x \cdot x^3 + 2018 \cdot e^x \cdot 3x^2 + \binom{2018}{2} e^x \cdot 6x + \binom{2018}{3} e^x \cdot 6$$

$$= e^x \cdot \left(x^3 + 6054 \cdot x^2 + \binom{2018}{2} \cdot 6x + 6 \cdot \binom{2018}{3} \right)$$

(c)

For an arbitrary sequence $(h_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}\setminus\{0\}$ with $\lim_{n\to\infty} h_n = 0$ use the hint:

null sequence   null sequence

$$f(x_0+h_n) = f(x_0) + h_n f'(x_0) + c_n h_n , \quad f(x_0-h_n) = f(x_0) - h_n f'(x_0) + d_n h_n ,$$

$$\frac{1}{2h_n} \cdot \left[f(x_0+h_n) - f(x_0-h_n) \right] = \frac{1}{2h_n} \cdot \left[2h_n \cdot f'(x_0) + h_n \cdot (c_n - d_n) \right]$$

$$= f'(x_0) + \tfrac{1}{2}(c_n - d_n) \xrightarrow{n\to\infty} f'(x_0) \qquad \square$$