Problem 3 *Applications of the mean value theorem* (4 points)
Let $a, b \in \mathbb{R}$, $a < b$ and $f\colon [a,b] \to \mathbb{R}$ be a continuous function that is differentiable on (a,b). Show:

a) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

b) If $f'(x) = f(x)$ for all $x \in (a,b)$, then there is a $c \in \mathbb{R}$ with $f(x) = ce^x$ for all $x \in [a,b]$.
Hint: Consider the function $x \mapsto f(x)e^{-x}$.

c) If $g : [a,b] \to \mathbb{R}$ is another continuous function that is differentiable on (a,b) with $g'(x) = f'(x)$ for all $x \in (a,b)$, then there is a $C \in \mathbb{R}$ with $g(x) = f(x) + C$.

d) For all $x, y \in \mathbb{R}$, we have $|\sin^3(x) - \sin^3(y)| \leq 3|x - y|$.

Solutions

(a) By mean value theorem (mvt): For all $x \in [a,b]$ there is a ξ:

$$\frac{f(x) - f(a)}{x - a} = f'(\xi) = 0 \quad (\text{by assumption})$$

$$\Rightarrow \quad f(x) = f(a) \quad \text{for all } x \in [a,b] \Rightarrow f \text{ constant}$$

(b) Define: $g(x) = f(x)e^{-x}$ (differentiable on (a,b))

$\overset{(\text{mvt})}{\Rightarrow}$ For all $x \in [a,b]$ there is a ξ with:

$$\frac{g(x) - g(a)}{x - a} = g'(\xi) = f'(\xi)e^{-\xi} - f(\xi)e^{-\xi}$$

$$= 0 \quad (\text{by assumption})$$

$$\overset{(a)}{\Rightarrow} \quad g \text{ constant} =: c \Rightarrow f(x) = ce^x \quad \text{for all } x \in \mathbb{R}.$$

(c) For all $x \in \mathbb{R}$, we have: $h(x) := g(x) - f(x)$
with $h'(x) = 0$ for all $x \in (a,b)$.

$$\overset{(a)}{\Longrightarrow} \; h \text{ constant} =: C \;\Longrightarrow\; g(x) = f(x) + C$$

for all $x \in \mathbb{R}$.

(d) $f(x) = \sin^3(x)$, $f'(x) = 3\sin^2(x) \cdot \cos(x)$

$$\sup_{\xi \in \mathbb{R}} |f'(\xi)| = 3$$

$$\overset{(\text{mvt})}{\Longrightarrow} \; |\sin^3(x) - \sin^3(y)| \leq 3 \cdot |x - y|$$