Problem 4 *Extreme points of a function* (2 bonus points)

Consider the function $f : (-1, 1) \to \mathbb{R}$ given by $f(x) := \mathrm{e}^{\frac{1}{\sqrt{x^2+1}}}$. Show that there is only one point x_0 where f has a local extremum. Does the function f attain its maximum?

Solutions

$$f(x) = e^{\frac{1}{\sqrt{x^2+1}}} \quad , \quad f'(x) = -\frac{1}{2}(x^2+1)^{-\frac{3}{2}} \cdot 2x \cdot e^{\frac{1}{\sqrt{x^2+1}}}$$

Critical points: $\quad f'(x) = 0$

$$\Longleftrightarrow \quad (x^2+1)^{-\frac{3}{2}} \cdot 2x = 0$$

$$\Longleftrightarrow \quad 2x = 0$$

$x = 0$ is the only critical point.

Since $f'(+\frac{1}{2}) < 0$ and $f'(-\frac{1}{2}) > 0$, we have a local maximum at $x = 0$, with $f(0) = e$.

The limits on the boundaries are:

$$f(\pm 1) = e^{\frac{1}{\sqrt{2}}} < e \quad \Longrightarrow \quad f(0) = e \text{ is the global maximum.}$$