Problem 1 *Taylor's formula and approximations* (4 bonus points)

a) Let $f\colon (0,\infty) \to \mathbb{R}$, $f(x) := \sqrt{x+1}$. Determine $T_{2,0}$, i.e., the Taylor polynomial of degree 2 for the function at expansion point $a = 0$, and the remainder term $R_{2,0}$.

b) Use part a) to show
$$\frac{155}{128} < \sqrt{\frac{3}{2}} < \frac{157}{128}\,.$$

c) Use Taylor's formula to find the first three digits after the decimal point for the decimal representation of $\sqrt{5}$. In order to do so, you should consider $f(x) = \sqrt{x+4}$ and the Taylor polynomial at expansion point $a = 0$.

d) Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) := \tanh(x)$. Determine $T_{6,0}$, i.e., the Taylor polynomial of degree 6 for the function at expansion point $a = 0$.

Solutions

(a) $f(x) = \sqrt{x+1}$, $f'(x) = \dfrac{1}{2\sqrt{x+1}}$, $f''(x) = -\frac{1}{2}\cdot\frac{1}{2}(x+1)^{-\frac{3}{2}}$

$f'''(x) = +\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(x+1)^{-5/2}$

$T_{2,0}(x) = 1 + \frac{1}{2}x + \frac{1}{2!}\cdot\left(-\frac{1}{4}\right)\cdot x^2 = 1 + \frac{1}{2}x - \frac{1}{8}\cdot x^2$

$R_{2,0}(x) = \frac{3}{8}(\xi+1)^{-5/2}\cdot\frac{1}{3!}x^3 = \frac{1}{16}(\xi+1)^{-5/2}\cdot x^3$ with $\xi \in (0,x)$

(b) $f\left(\frac{1}{2}\right) = \sqrt{\frac{3}{2}}$

$$T_{2,0}\left(\tfrac{1}{2}\right) = 1 + \frac{1}{4} - \frac{1}{32} = \frac{39}{32} = \underline{\frac{156}{128}}$$

$$\left|R_{2,0}\left(\tfrac{1}{2}\right)\right| = \frac{1}{16} \cdot \frac{1}{(\xi+1)^{5/2}} \cdot \frac{1}{8} < \frac{1}{16} \cdot \frac{1}{1} \cdot \frac{1}{8} = \underline{\frac{1}{128}} =: C_1$$

Hence:

$$T_{2,0}\left(\tfrac{1}{2}\right) - C_1 < f\left(\tfrac{1}{2}\right) < T_{2,0}\left(\tfrac{1}{2}\right) + C_1$$

$$\Longleftrightarrow \quad \frac{155}{128} < \sqrt{\frac{3}{2}} < \frac{157}{128}$$

(Since the remainder is here positive, the left estimate could also be chosen as $T_{2,0}\left(\frac{1}{2}\right) = \frac{156}{128}$)

(c) Same again: $f(x) = \sqrt{x+4}$, $f'(x) = \frac{1}{2}(x+4)^{-1/2}$, $f''(x) = -\frac{1}{4}(x+4)^{-3/2}$

$f'''(x) = +\frac{3}{8}(x+4)^{-5/2}$, $f^{(4)}(x) = -\frac{15}{16}\cdot(x+4)^{-7/2}$

Try Taylorpolynomial of order 3:

$$R_{3,0}(1) = -\frac{15}{16} \cdot \frac{1}{4!}(\xi+4)^{-7/2} \qquad \text{with} \quad \xi \in (0,1).$$

$$\Rightarrow |R_{3,0}(1)| < \frac{15}{2^4} \cdot \frac{1}{4!}\, 2^{-7} = \frac{1}{2^{10}} \cdot \frac{15}{2} \cdot \frac{1}{4\cdot 3\cdot 2} = \frac{1}{2^{14}} \cdot 5$$

$$< \frac{5}{16\,000} = 0.000\underline{3}125 < \underline{3}2 \cdot 10^{-5}$$

$$T_{3,0}(1) = 2 + \frac{1}{4}\cdot 1^1 - \frac{1}{64}\cdot 1^2 + \frac{1}{512}\cdot 1^3 = \frac{1145}{512} = 2.236\underline{33}$$

safe digits!

$$\underline{\sqrt{5} = 2.236\ldots}$$

(d) Calculate derivatives or using the Cauchy product!

$$f(x) = \tanh(x) \quad , \quad f'(x) = 1 - \tanh^2(x)$$

$$f''(x) = -2\tanh(x)\cdot(1-\tanh^2(x)) = -2f(x) + 2f(x)^3$$

$$f'''(x) = -2\cdot f'(x) + 6\cdot f(x)^2\cdot f'(x)$$

$$f^{(4)}(x) = -2\cdot f''(x) + 12\cdot f(x)\, f'(x)^2 + 6\cdot f(x)^2\cdot f''(x)$$

$$f^{(5)}(x) = -2\cdot f'''(x) + 12\cdot f'(x)^3 + 12\cdot f(x)\cdot 2f'(x)\cdot f''(x)$$
$$+12\cdot f(x)\cdot f'(x)\cdot f''(x) + 6\cdot f(x)^2\cdot f'''(x)$$

$\tanh(x)$ is an odd function $\Rightarrow$ Taylorpolynomial at expansion point $a=0$ is also odd polynomial.

$$\Rightarrow f(0) = 0\,,\; f'(0) = 1,\; f''(0) = 0,\; f'''(0) = -2$$

$$f^{(4)}(0) = 0 \quad , \quad f^{(5)}(0) = +4 + 12 = 16$$

$$\Rightarrow T_{6,0}(x) = x - \frac{1}{3!}\cdot 2\cdot x^3 + \frac{1}{5!}\cdot 16\cdot x^5 \quad = x - \frac{x^3}{3} + \frac{2}{15}x^5$$