Problem 4 *Proofs* (4 bonus points)
Let $a, b \in \mathbb{R}$ with $a < b$.

a) Show that each monotonically increasing function $f : [a, b] \to \mathbb{R}$ is Riemann-integrable.

b) Let $g : [a, b] \to [0, \infty)$ be a continuous function (and hence Riemann-integrable) with

$$\int_a^b g(x)\, dx = 0 \, .$$

Show that $g(x) = 0$ holds for all $x \in [a, b]$.

Solutions

(a) Let $\varepsilon > 0$ and $x_1 < x_2 < \dots < x_N$ be a partition of $[a,b]$ with $x_0 := a$ and $x_N := b$, and $x_{i+1} - x_i = \varepsilon$.

The lower sum:

$$L = \sum_{i=1}^{N} (x_i - x_{i-1}) \cdot \underbrace{\inf\{f(x) \mid x \in [x_i, x_{i-1}]\}}_{\substack{f(x_i) \text{ since } f \text{ is} \\ \text{mon. increasing}}}$$

The upper sum:

$$U = \sum_{i=1}^{N} (x_i - x_{i-1}) \cdot \underbrace{\sup\{f(x) \mid x \in [x_i, x_{i-1}]\}}_{\substack{f(x_{i+1}) \text{ since } f \text{ is} \\ \text{mon. increasing}}}$$

The difference is:

$$U - L = \sum_{i=1}^{N} (x_i - x_{i-1}) \cdot \left[f(x_i) - f(x_{i-1})\right]$$

$$= \varepsilon \cdot \left(\sum_{i=1}^{N} f(x_i) - \sum_{i=1}^{N} f(x_{i-1}) \right) \quad \text{(telescoping)}$$

$$= \varepsilon \cdot \big(\underbrace{f(b) - f(a)}_{\text{constant}} \big) =: \varepsilon'$$

For each $\varepsilon' > 0$, there is partition such that $U - L < \varepsilon'$.

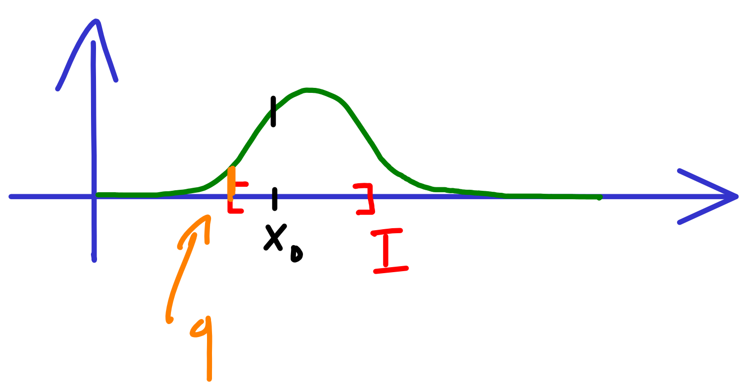
$\overset{\text{tutorial}}{\Longrightarrow}$ f is Riemann-integrable.

(b) Assume there is an $x_0 \in [a,b]$ with $g(x_0) > 0$.

Since g is continuous, there is a closed interval $I \subseteq [a,b]$ with $x_0 \in I$ such that $g(x) > 0$ for all $x \in I$.

(Provable with the ε-δ-criterion for continuity.)

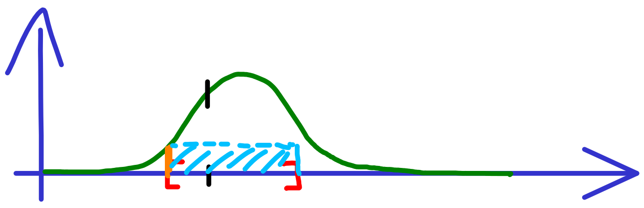
Then: $\inf\limits_{x \in I} g(x) = \min\limits_{x \in I} g(x) =: q > 0$



We set $\delta > 0$ for the length of I.

For each partition P that includes the interval I, we get for the lower sum (that is non-negativ!):

$$L_P \geq \delta \cdot \inf_{x \in I} g(x) = \delta \cdot q > 0$$



This means:

$$0 < L_P \leq \int_a^b g(x)\, dx .$$

By contraposition, we have proven the claim.