The Bright Side of Mathematics

Problem 1: For which x are the following functions defined and differentiable?

Calclulate their respective derivatives.

(a) $f_1(x) = \dfrac{x^2 - e^{(x^2)}}{x^2 + (e^x)^2}$

(b) $f_2(x) = \log(\cos(x))$

(c) $f_3(x) = \arcsin(x) \cdot a^x$ for $a > 0$

$f_1(x) = \dfrac{x^2 - e^{(x^2)}}{\underbrace{\underbrace{x^2}_{\geq 0} + \underbrace{(e^x)^2}_{>0}}_{>0}}$

$f_1 : \mathbb{D}_1 \to \mathbb{R}$

$\mathbb{D}_1 = \mathbb{R}$

$(x^n)' = nx^{n-1}$ $n \in \mathbb{N}$

$f_1(x) = \dfrac{u(x)}{v(x)}$ Quotient rule: $f_1'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$

with $u(x) = \overbrace{x^2}^{A_1} - \overbrace{e^{(x^2)}}^{A_2}$

$v(x) = x^2 + (e^x)^2$

$u'(x) = 2x - e^{(x^2)} \cdot 2x$ (Chain rule)

$v'(x) = 2x + 2e^x \cdot e^x$

$= 2x + 2e^{2x}$

$$f_1'(x) = \frac{[2x - e^{(x^2)} \cdot 2x][x^2 + (e^x)^2] - [x^2 - e^{(x^2)}][2x + 2e^{2x}]}{[x^2 + (e^x)^2]^2}$$

Problem 2: Show that the derivative of an even function $f : \mathbb{R} \to \mathbb{R}$ is odd.

Show that the derivative of an odd function $f : \mathbb{R} \to \mathbb{R}$ is even.

Problem 3: Find a rational number $x \in \mathbb{Q}$ such that $\left| x - \sqrt{\frac{3}{2}} \right| < 0.003$.

Problem 4: Let $a, b \in \mathbb{R}$, $a < b$, $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function.

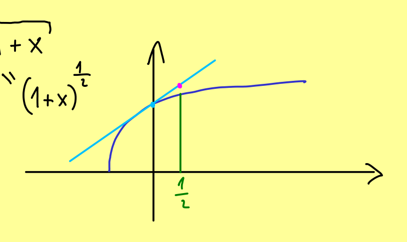
Show:

(a) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

(b) If $f'(x) = f(x)$ for all $x \in (a,b)$, then $f(x) = ce^x$ for a constant $c \in \mathbb{R}$

Problem 3: Find a rational number $x \in \mathbb{Q}$ such that $\left| x - \underbrace{\sqrt{\frac{3}{2}}}_{1 \leq \text{irrational} \leq 2} \right| < 0.003$.

Taylor polynomial: $f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$



$f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$, $f'(0) = \frac{1}{2}$

$f''(x) = \frac{1}{2} \cdot \frac{1}{2} \cdot (-1) \cdot (1+\underset{=0}{x})^{-\frac{3}{2}}$, $f''(0) = -\frac{1}{4}$

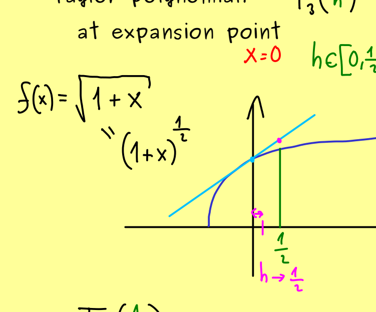
$f'''(x) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} (-1)^2 (1+x)^{-\frac{5}{2}}$, $f'''(0) = \frac{1}{2^3} \cdot 3$

$f^{(4)}(x) = \frac{1 \cdot 3 \cdot 5}{2^4} \cdot (-1)^{4-1} (1+x)^{-\frac{7}{2}}$ $\rightsquigarrow$ general formulation possible

Taylor polynomial: $T_3(h) = f(0) + \frac{1}{1!} f'(0) \cdot h^1 + \frac{1}{2!} f''(0) \cdot h^2 + \frac{1}{3!} f'''(0) \cdot h^3$

at expansion point $x = 0$ $h \in [0, \frac{1}{2}]$

$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}}$



$T_3\left(\frac{1}{2}\right) = 1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2^2}\right) \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{\cancel{3} \cdot 2} \cdot \frac{\cancel{3}}{2^3} \cdot \left(\frac{1}{2}\right)^3$

$= 1 + \frac{1}{2^2} - \frac{1}{2^5} + \frac{1}{2^7} = \frac{2^7 + 2^5 - 2^2 + 1}{2^7} = \frac{\overbrace{128 + 32}^{160} - 4 + 1}{128}$

$= \frac{157}{128} = 1.2265625 \approx \sqrt{\frac{3}{2}} \pm 0.003$

safe two digits?

$(0.003 >) \left| T_3\left(\frac{1}{2}\right) - \sqrt{\frac{3}{2}} \right| = \left| f\left(\frac{1}{2}\right) - T_3\left(\frac{1}{2}\right) \right| = \left| R_3\left(\frac{1}{2}\right) \right| \leq \dots < 0.003$

$R_3\left(\frac{1}{2}\right) = \frac{1}{4!} f^{(4)}(\xi) \cdot \left(\frac{1}{2}\right)^4$, $\xi \in [0, \frac{1}{2}]$

$\left| R_3\left(\frac{1}{2}\right) \right| = \frac{1}{4 \cdot \cancel{3} \cdot 2} \cdot \frac{1 \cdot \cancel{3} \cdot 5}{2^4} \cdot \underbrace{\left| \frac{1}{(1+\xi)^{\frac{7}{2}}} \right|}_{\leq 1} \left(\frac{1}{2}\right)^4 \leq \frac{5}{2^2 \cdot 2} \cdot \frac{1}{2^4} \cdot \frac{1}{2^4}$

$\xi \to 0$ worst case

$\leq 5 \cdot \frac{1}{2^{11}} = \frac{5}{2048} \leq \frac{5}{2000} = 0.0025 < 0.003$

$$\boxed{1.2235\ldots \leq \sqrt{\frac{3}{2}} \leq 1.2295\ldots}$$

Problem 2:

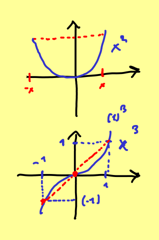
$f : \mathbb{R} \to \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R}$

f even: $f(x) = f(-x)$ $x \in \mathbb{R}$

f odd: $\boxed{f(x) = -f(-x)}$ $x \in \mathbb{R}$

Counter Example: $(\underbrace{1 + x^3}_{\text{no symmetry}})' = 3x^2$ even



Problem 2: a) Show that the derivative of an even function $f : \mathbb{R} \to \mathbb{R}$ is odd.

b) Show that the derivative of an odd function $f : \mathbb{R} \to \mathbb{R}$ is even.

$\sigma : \mathbb{R} \to \mathbb{R}$, $x \mapsto (-x)$ is diff'ble $\sigma'(x) = -1$

f even: $f(x) = f(\sigma(x)) = f \circ \sigma(x)$ (I)

f odd: $f(x) = \sigma \circ f \circ \sigma(x)$

a) $f'(x) \underset{\text{even}}{=} (f \circ \sigma)'(x) \underset{\text{chain rule}}{=} f'(\sigma(x)) \cdot \sigma'(x) \underset{\text{def. of } \sigma}{=} f'(-x) \cdot (-1)$

b) $f'(x) \underset{\text{odd}}{=} (\sigma \circ f \circ \sigma)'(x) \underset{\text{chain rule}}{=} \sigma'(f \circ \sigma(x)) \cdot (f \circ \sigma)'(x) = (-1) f'(-x) \cdot (-1)$

$= f'(-x)$

Problem 4: Let $a, b \in \mathbb{R}$, $a < b$, $f : [a,b] \longrightarrow \mathbb{R}$ be a differentiable function.

Show: mean value theorem

(a) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

(b) If $\boxed{f'(x) = f(x)}$ for all $x \in (a,b)$, then $f(x) = ce^x$ for a constant $c \in \mathbb{R}$

(b) $g(x) := f(x) \cdot e^{-x}$, $g'(x) \overset{\text{product rule}}{=} \underbrace{f'(x)}_{= f(x)} \cdot e^{-x} + f(x) \cdot (-1) e^{-x}$

$= 0$ ✓

$\overset{(a)}{\Longrightarrow} g$ constant $= c \in \mathbb{R}$

$\Longrightarrow f(x) = c \cdot e^x$